RAN-2392

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M.A. (Part II) Maths Examination

March / April - 2019

Mathematics

(Special Functions-I)

સૂચના : / Instructions

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fill up strictly the details of ☛ signs on your answer book

Name of the Examination:

☛ **M.A. (Part II) Maths**

Name of the Subject :

☛ **Mathematics**

Subject Code No.: 2 3 9 2

Seat No.:

Student's Signature

(1) Attempt all questions.
(2) Figures to the right indicate marks.
(3) Follow the usual notations and conventions.

Q.1 **Answer the following questions**

(I) If $Z > 0$ and γ be Euler constant then in usual notation prove that **07**

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+1)}$$

(II) If n is an integer and $\mathrm{Re}(Z) > 0$, show that $\Gamma(z) = \lim_{n \to \infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt$ **07**

(III) With usual notation show that $\Gamma(z) = \lim_{n \to \infty} \frac{n!\, n^z}{z(z+1)(z+2) \ldots\ldots (z+n)}$

OR

Q.1 (I) Define Euler constant γ. Show that it is less that unity. **07**

(II) In usual notation prove that $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$; p, q > 0. **07**

(III) If $0 \leq t < n$, n be a positive integer then show that

$$0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2 e^{-t}}{n}$$

Q.2 **Answer the following questions**

(I) If $|z| < 1 \; and \; |1-z| < 1, \operatorname{Re}(c) > 1, \operatorname{Re}(c-a-b) > 0,$ **07**

a, b, c, c-a, c-b, c-a-b are not integer then prove that

$$F(a,\, b;\, a+b+1-c;\, 1-z) = \frac{\Gamma(a+b+1-c)\,\Gamma(1-c)}{\Gamma(a+1-c)\,\Gamma(b+1-c)}\; F(a,b;\, c;z)$$

$$+$$

$$\frac{\Gamma(a+b+1-c)\,\Gamma(c-1)}{\Gamma(a)\,\Gamma(b)}\; z^{1-c}\, F(a+b+1-c, b+1-c; 2-c; z)$$

(II) Show that the general Sol^n of the hypergeometric differential equation **07**

z(1-z)w'' + [c-(a+b+1)z)]w' – abw = o valid for $|1-z| < 1$ is given by

w = A F (a, b ; a+b-c+1; 1-z) +B (1-z)$^{c-a-b}$ F (c-b, c-a, 1+c-a-b; 1-z)

(III) If $|y| < \frac{1}{2}$ and $\left|\frac{y}{1-y}\right| < 1$, show that **06**

OR

$$(1-y)^{-a}\, F\left[\begin{matrix} \frac{a}{2}, \frac{a}{2}+\frac{1}{2} \\ b+\frac{1}{2} \end{matrix} ;\, \frac{y^2}{(1-y)^2}\right] = F\left[\begin{matrix} a, b \\ 2b \end{matrix} ;\, 2y\right]$$

Q.2.

(I) Derive the differential equation for the Hypergeometric function *F(a, b ; c ; z)*. **07**

(II) With usual notation prove that

$[a+(b-c)z]\, F = a\,(1-z)\, F(a+) - c^{-1}\,(c-a)(c-b) z F(c+)$ **07**

$(1-z)\, F = F\,(a-) - c^{-1}\,(c-b) z F(c+)$

$(1-z)\, F = F\,(b-) - c^{-1}\,(c-a) z F(c+)$

(III) Derive the integral form of the Hypergeometric function *F(a, b; c; z)* **06**
And hence obtain F (a, b; c; 1)

Q.3 **Answer the following questions**

(I) With usual notation show that **07**

$$P_n(x) = \frac{2^n \left(\frac{1}{2}\right)_n (x-1)^n}{n!}\; {}_2F_1\left(-n, -n; -2n; \frac{2}{1-x}\right)$$

(II) Prove that. $\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$ **07**

(III) Prove that $\sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n = e^{xt} J_0(t\sqrt{1-x^2})$ **06**

OR

Q.3

(I) If-1 <x<1 and if *n* is any integer show that $|P_n(x)| < \left[\frac{\pi}{2n(1-x^2)}\right]^{\frac{1}{2}}$ **07**

(II) For non negative integral *n* show that **07**

$$x^n = \frac{n!}{2^n} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(2n-4k+1)P_{n-2k}(x)}{k!\left(\frac{3}{2}\right)_{n-k}}$$

(III) Derive the Rodrigue's formula for P_n (x). **06**

Q.4 **Answer the following questions**

(I) Obtain the relations **07**

$xH_n(x) = nH'_{n-1}(x) + nH_n(x)$, $H_n(x) = 2xH_{n-1}(x) - nH'_{n-1}(x)$

(II) With usual notation prove that **07**

$$\sum_{n=0}^{\infty} \frac{H_n(x)\, H_n(y)^n}{n!} = (-4t^2)^{-\frac{1}{2}} \exp\left[y^2 - \frac{(y-2xt)^2}{1-4t^2}\right]$$

(III) Show that $\int_0^{\infty} \exp(1-x^2) H_{2S}(x) H_{2s+1}(x)\,dx$ **06**

$$= \frac{(-1)^{k+1} 2^{2k+2s} \left(\frac{1}{2}\right)_k \left(\frac{3}{2}\right)_s}{(2s+1-2k)}$$

OR

Q.4 (I) Prove that $\sum_{n=0}^{\alpha} \frac{H_{n+k}(x)t^n}{n!} = \exp(2xt - t^2)\, H_k(x - t)$ **07**

(II) With usual notation prove that **07**

$$\sum_{n=0}^{\infty} \frac{(c)_n H_n(x) t^n}{n!}$$

$$=(1-2xt)^{-c} \; {}_2F_0 \left[\begin{matrix} \frac{c}{2}, \frac{c}{2} + \frac{1}{2} \; ; & \\ & \frac{-4t^2}{(1-2xt)^2} \\ ---- \; ; & \end{matrix} \right]$$

(III) With usual notation prove that **06**

$$H_{2n}(0) = (-1)^n 2^{2n} \left(\frac{1}{2}\right)_2 \qquad H'_{2n+1}(0) = (-1)^n 2^{2n+1} \left(\frac{n}{2}\right)_n$$

Q.5 **Answer the following questions**

(I) If $\{\phi_n(x)\}$ is a simple set of polynomials and if p(x) is a polynomial of degree m, then show that there exists constant c_k such that **07**

$p(x) = \sum_{k=0}^{m} c_k\, \phi_k(x)$, where are functions of k and of any parameters involved in $p(x)$.

(II) Let $\{\phi_n(x)\}$ is a simple set of real polynomials orthogonal with respected to w(x) > 0 on a < x < b. Let h_n be the the leading coefficient in $\Phi_n(x)$ so that $\Phi_n(x) = h_n x^n + \pi_{n-1}$ where π_{n-1} is a polnomial of degree n-1. Let $g_k = (\phi_k, \phi_k)$ then prove that **07**

$$\sum_{k=0}^{n} gk^{-1} \phi_k(x) \phi_k(y) = \frac{h_n}{g_n h_{n-1}} \left[\frac{\phi_{n+1}(y)\, \phi_n(x) - \phi_{n+1}(x) \phi_n(y)}{y - x} \right]$$

(III) Prove that **06**

$$H_n(x) = \frac{n!}{\pi} \int_0^{\pi} exp(2x\cos\theta - \cos 2\theta) \cos(2x \sin\theta - \sin 2\theta - n\theta)\, d\theta$$

Q.5 Answer the following questions

(I) Define uniform convergence of the infinite prefect. If for a positive constant M_n such that $\sum_{n=1}^{\alpha} M_n$ is convergent and $|a_n(k)| < M_n$ for all z the product $\prod_{n=1}^{\infty} (1+a_n)$ is uniformly convergent. **07**

(II) Define absolute convergence of an infinite product. Show that the product $\prod_{n=1}^{\infty} (1+a_n)$ with zero factor deleted is absolutely convergent iff $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. **07**

(III) It Z is not a negative integer then show that $\lim_{n\to\infty} \frac{(n-1)!n^z}{(z+1)(z+2)......(z+n-1)}$ is exists. **06**